Basic Set Theory

A set is a collection of elements. (This not a precise definition, since I don’t say what the words “collection” and “element” mean.)

The fundamental property of sets is: Two sets are deemed to be equal if and only if they have the same elements.

Sets are different from lists in two important ways:

- repetition of elements doesn’t matter.
- order of elements doesn’t matter.

For example, \( \{1, 2, 3\} = \{3, 1, 3, 2, 1\} \). These define the same set (they have the same elements), but when regarded as lists they are quite different.

There are inherent difficulties with this naive concept of set (look up Russell’s paradox). Rather than allowing a set to be any collection of elements, in order to avoid paradoxes we should only allow collections which are not “too big” in a certain sense.

Elements: If \( A \) is a set and \( x \) is one of its elements then we write \( x \in A \). One says that \( x \) is in the set \( A \), or that \( x \) belongs to \( A \), or that \( x \) is a member of \( A \).

Non-elements: If \( x \) is not an element in the set \( A \) then we write \( x \notin A \).

Definition (Empty set). The empty set (also called the null set) is the set with no elements. This set is written as \( \{ \} \) or as \( \emptyset \).

Elements can be tricky, especially since we allow them to be other sets! For example, the elements of the set \( B = \{\{1\}, \{2\}, \{3, 4\}\} \) are the sets \( \{1\}, \{2\}, \) and \( \{3, 4\} \). For the set \( B \), it is true that \( 1 \notin B, 2 \notin B, 3 \notin B, \) and \( 4 \notin B \). In fact, there are no numbers at all in \( B \), since all the members of \( B \) are sets.

Listing sets: Often we write a set by listing its elements (in some order, which doesn’t matter). Thus \( A = \{2, 5, 1, 9\} \) is the set consisting of the elements \( 1, 2, 5, 9 \). We already did this in example on previous slides.
For infinite sets we sometimes use the \ldots notation to indicate that a given pattern continues indefinitely. For example, the set \{1, 3, 5, 7, 9, \ldots\} is the set of all odd counting numbers and the set \{0, \pm2, \pm4, \pm6, \ldots\} is the set of all even integers.

\textit{Standard sets of numbers:} The following notations have become the standard for various common sets of numbers used throughout mathematics.

- \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} = \text{natural numbers}
- \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \text{integers}
- \mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\} = \text{rational numbers}
- \mathbb{R} = \text{real numbers}
- \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} = \text{complex numbers}

where it is understood that \(i^2 = -1\). (The number \(i\) is called the imaginary unit. Warning: Physicists and some chemists use the symbol \(j\) instead of \(i\) for the imaginary unit.)

\textit{Set-builder notation:} We will often build new sets from existing ones by using a condition on its elements. If \(P(x)\) is a statement about \(x\) and if \(A\) is a given set then either of the equivalent notations

\[
\{x \in A : P(x)\} = \{x \in A \mid P(x)\}
\]

means (by definition) the set of all \(x\) in the set \(A\) for which \(P(x)\) is true. (We already used this notation on the previous slide!)

For example, \(\{x \in \mathbb{R} : 1 \leq x \leq 3\}\) defines the closed interval \([1, 3]\) in the real line. We can define the set of odd natural numbers as \(\{x \in \mathbb{N} : x = 2k + 1 \text{ for some } k \in \mathbb{N}\}\). For another example of this notation, the set \(\{x \in \mathbb{R} : x^2 - 4 = 0\}\) is the set of all real numbers \(x\) satisfying the equality \(x^2 - 4 = 0\). As we know, this is just the set \(\{2, -2\}\).

We now consider the basic relations and operations on sets.

\textbf{Definition (Subsets).} We write \(A \subset B\) (or \(A \subseteq B\)) if every element of \(A\) is also an element of \(B\). We can also write this as \(B \supset A\) (or \(B \supseteq A\)). When \(A \subset B\) we say that \(A\) is a \textit{subset}\ of \(B\) or that \(B\) \textit{contains} \(A\).
Warning: Note that the statements $A \subset B$ and $A \in B$ do NOT have the same meaning. Note also that $A \subset A$: every set is contained in itself, and every set contains the empty set: $\emptyset \subset A$.

**Definition** (Proper subsets). If $A \subset B$ but $A \neq B$ then we will write $A \varsubsetneq B$. In this case we say that $A$ is a proper subset of $B$, or that $A$ is strictly contained within $B$.

**Proposition** (Set equality). For sets $A, B$ we have $A = B$ if, and only if, $A \subset B$ and $B \subset A$.

In words: two sets are equal if and only if each is contained in the other.

This is often used in proofs to show equality of two sets. In other words, to prove that $A = B$, you have to prove two things: that $A \subset B$ and $B \subset A$.

We will see examples later.

**Definition** (Union of sets). The union or join of two given sets $A, B$ is the set $A \cup B$ whose elements are obtained by collecting together all the elements in the two sets.

In other words, we can write the definition of union formally as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$  

**Definition** (Intersection of sets). The intersection or meet of two given sets $A, B$ is the set $A \cap B$ of elements the two sets have in common.

In other words, we can write the definition of intersection formally as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$  

For example, if $A = \{1, 3, 5\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{3\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.  

**Definition** (Complements). If \( B \subset A \) the complement of \( B \) is the set

\[ B^c = \{ x \in A : x \notin B \}. \]

In words, it is the set of all elements of \( A \) which are not elements of \( B \). This can also be written as \( A - B \) or \( A \setminus B \).

In fact, it is not necessary that \( B \) is a subset of \( A \). For any sets \( A, B \) we can still define the complement of \( B \) in \( A \) to be

\[ A - B = \{ x \in A : x \notin B \}. \]

*Exercise:* If you have been following along, you should be able to show that \( A - B = A \cap B^c \).

**Definition** (Product of two sets). Let \( A, B \) be two given sets. Write \( A \times B \) for the set of all ordered pairs \((x, y)\) such that \( x \in A \) and \( y \in B \). This construction should look familiar; it is called *Cartesian product* or *direct product*.

Formally, we have \( A \times B = \{ (x, y) \mid x \in A, y \in B \} \).

**Definition** (Product of many sets). If \( A_1, A_2, \ldots, A_n \) are given sets then we can form their product \( A_1 \times A_2 \times \cdots \times A_n \), the elements of which are called ordered \( n \)-tuples. Formally, the definition is:

\[ A_1 \times \cdots \times A_n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in A_i \text{ for all } i = 1, \ldots, n \}. \]

The special case \( A \times A \times \cdots \times A \) (with \( n \) factors) is written as \( A^n \). For example, \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \} \).