Well-ordering property and induction

1. Show that \( \mathbb{Z} \) does not satisfy the well-ordering property: i.e., that it is \emph{not} the case that every nonempty subset of \( \mathbb{Z} \) has a least element.

2. Let \( \mathbb{Q}_{>0} = \{ x \in \mathbb{Q} : c > 0 \} \) be the set of positive rational numbers. Show that \( \mathbb{Q}_{>0} \) does not satisfy the well-ordering principle: i.e., that it is \emph{not} the case that every nonempty subset of \( \mathbb{Q}_{>0} \) has a least element.

3. Prove by induction that for all \( n \geq 0 \) we have \( \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1} \).

4. Prove by induction that \( n^2 < 2^n \) for all \( n \geq 5 \).

5. Prove by induction that \( \sum_{j=0}^{n} j = \frac{n(n + 1)}{2} \) for all \( n \geq 0 \).

6. Prove by induction that \( \sum_{j=0}^{n} j^2 = \frac{n(n + 1)(2n + 1)}{6} \) for all \( n \geq 0 \).

7. Prove by (S)POMI that \( (x - y) \) is a factor of \( x^n - y^n \) for all \( n \geq 1 \).

8. Prove by induction that for any \( n \geq 2 \) propositions \( P_1, P_2, \ldots, P_n \) the proposition
   \[
   P_1 \Rightarrow (P_2 \Rightarrow (\cdots \Rightarrow (P_{n-1} \Rightarrow P_n)))
   \]
   is false for exactly one choice of truth values for the \( P_i \). Your proof should also indicate what this truth assignment is.

9. Prove by induction that for any \( n \geq 2 \) propositions \( P_1, P_2, \ldots, P_n \) and any \( Q \), the implication
   \[
   (P_1 \text{ and } P_2 \text{ and } \cdots \text{ and } P_n) \Rightarrow Q
   \]
   is logically equivalent to
   \[
   P_1 \Rightarrow (P_2 \Rightarrow (\cdots \Rightarrow (P_n \Rightarrow Q)))
   \]
   To ease notation, you may use the notation \( R \sim S \) to indicate that two propositions \( R \) and \( S \) are logically equivalent.

10. The Fibonacci sequence is defined by
    \[
    F_0 = 0 \\
    F_1 = 1 \\
    F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.
    \]
As we will see the sequence is intimately connected with \( \alpha = \left( \frac{1 + \sqrt{5}}{2} \right) \) and \( \bar{\alpha} = \left( \frac{1 - \sqrt{5}}{2} \right) \). These numbers are both roots of the polynomial \( x^2 - x - 1 \), which means they both satisfy the relation \( x^2 = x + 1 \). This will simplify your algebra below. The number \( \alpha \) is called the golden ratio and has quite an interesting cultural history touching on philosophy, architecture, painting, and music.

(a) Prove by induction that \( F_n \geq \alpha^{n-2} \) for all \( n \geq 2 \). You may use a decimal approximation to cover the base cases \( F_2 \) and \( F_3 \).

(b) Prove by induction that for all \( n \geq 0 \) the \( n \)-th Fibonacci number can be expressed as

\[
F_n = \frac{1}{\sqrt{5}} (\alpha^n - \bar{\alpha}^n).
\]

11. Define \( S_n = \sum_{i=1}^{n} \frac{1}{i} \). Prove by induction that \( S_{2n} \geq 1 + \frac{2}{n} \) for all \( n \geq 0 \).

12. Show that for any \( n \geq 24 \) a postage of \( n \) cents can be obtained using 5- and 7-cent stamps.

13. Consider a configuration of \( n \) lines in the plane satisfying:

- Any two lines intersect in a point.
- No three lines intersect in a point.

Show that for all \( n \geq 1 \) any such configuration of \( n \) lines divides the plane into \( \frac{n(n+1)}{2} + 1 \) different regions.

HINT: first draw examples of such configurations for \( n = 1, 2, 3 \), just to see what’s going on. For the induction step, observe that to go from a configuration of \( n \) lines to \( n + 1 \) lines we simply lay an additional line over all the other ones in a careful way. How many new regions do you get?

14. Consider the following game. There are two players, Player 1 and Player 2, and two piles of beads, each with \( n \) beads. On a given turn a player chooses a pile and picks up some (nonzero) amount of beads from that pile. The last player to pick up a bead wins. Player 1 goes first.

Describe a strategy for Player 2 that ensures a win, no matter what \( n \) is. Use (S)POMI to prove your claim.

15. Prove by (S)POMI that a \( 2^n \times 2^n \) chessboard with one piece missing can be tiled using L-shaped pieces, where each L-shaped piece covers three squares.
16. Assuming POMI is true, show that SPOMI is true. More precisely, assume $A \subset \mathbb{Z}^+$ satisfies the two properties of SPOMI and use POMI to show that $A = \mathbb{Z}^+$.

17. Assuming SPOMI is true, show that the well-ordering property is true. In other words, prove using strong induction that if $A \subset \mathbb{Z}^+$ is a nonempty set, then $A$ has a least element. [Hint: if $1 \in A$, then clearly it has a least element; if not, consider the set $B = \mathbb{Z}^+ - A$ of all elements not in $A$.]