Logic and proof method

1. Compute the truth table of \([(\neg P) \lor (\neg Q)]\). Your table should have 5 columns: namely, \(P\), \(Q\), \(\neg P\), \(\neg Q\), \(\neg P \lor \neg Q\).

2. Compute the truth table of \([(\neg P) \land (\neg Q)]\). Again, your table should have 5 columns. Compare with the truth table of \([\neg (P \lor Q)]\).

3. Show that \(P \Rightarrow Q\), and \((\neg Q) \Rightarrow (\neg P)\) have the same truth table. More specifically, make a truth table whose columns are \(P\), \(Q\), \(P \Rightarrow Q\), and \((\neg Q) \Rightarrow (\neg P)\).

4. Compute the truth table of \(P \Rightarrow (Q \Rightarrow R)\).

5. “When I had money, I had plenty friends. I ain’t got no money baby, honey even no friends.”-Mississippi Fred McDowell. Let \(P\) be the proposition ‘I have money’, and let \(Q\) be the proposition ‘I have friends’.
   a. Express the following sentences in terms of \(P\), \(Q\) and the logical connectives ‘\(\Rightarrow\)’, ‘\(\iff\)’, and ‘\(\neg\)’:
      (i) I have friends only if I have money.
      (ii) In order for me to have friends it is sufficient that I have money.
      (iii) For I to have no money, it is necessary that I have no friends.
      (iv) I have friends if and only if I have money.
   b. For each possible combination of having or not having money, and having or not having friends, explain whether the proposition in (iii) is true or false.
   c. Which of items (i)-(iv) do you think Mr. McDowell had in mind?

6. Is the sentence ‘This sentence is false’ a proposition?

7. Show that \(P \Rightarrow (Q \Rightarrow (R \Rightarrow (S \Rightarrow T)))\) is false for exactly one choice of truth values for \(P, Q, R, S, T\). (Don’t compute the whole truth table!) Can you prove the generalization to the implication \(P_1 \Rightarrow (P_2 \Rightarrow (\cdots \Rightarrow (P_{n-1} \Rightarrow P_n)\cdots))\)?

8. Show that the three propositions
(i) \( P \Rightarrow (Q \Rightarrow R) \)
(ii) \((P \Rightarrow Q) \Rightarrow R\)
(iii) \((P \text{ and } Q) \Rightarrow R\)

are all logically equivalent.

9. Prove the De Morgan’s Laws. That is, show that

(a) Not \((P \text{ or } Q)\) is equivalent to \([(\text{Not } P) \text{ and } (\text{Not } Q)]\), and
(b) Not \((P \text{ and } Q)\) is equivalent to \([(\text{Not } P) \text{ or } (\text{Not } Q)]\).

10. Suppose \(\mathcal{P}, Q\) and \(R\) are three compound logical proposition. Prove that if \(\mathcal{P}\) is equivalent to \(Q\), and \(Q\) is equivalent to \(R\), then \(\mathcal{P}\) is equivalent to \(R\).

11. Suppose \(\mathcal{P}\) and \(Q\) are two compound propositions built up from the simple propositions \(P_1, P_2, \ldots, P_n\) using our logical connectives. Show that \(\mathcal{P}\) and \(Q\) are logically equivalent if and only if \(\mathcal{P} \iff Q\) is a tautology. Hint: imagine making the truth table with rows for each of \(P_1, P_2, \ldots, P_n\), and with columns for \(\mathcal{P}, Q,\) and \(\mathcal{P} \iff Q\).

12. Let \(n\) be an integer. Prove by contradiction that \(n\) or \(n + 1\) is odd.

13. The direct method of proving a disjunction of the form \(P\) or \(Q\) is true is to show that \(P\) is true or \(Q\) is true.

(a) Show that \((P \text{ or } Q)\) is logically equivalent to \(\neg P \Rightarrow Q\).
(b) Use the result in (a) to come up with an alternative to the direct method of proving a disjunction of the form \(P\) or \(Q\).

14. We saw above that \((P \text{ or } Q)\) is logically equivalent to \(\neg P \Rightarrow Q\).

(a) Explain how this allows us to do without ‘or’ as a logical connective. In other words show that for any compound proposition \(\mathcal{P}\) using the connectives ‘or’, ‘and’, ‘\(\Rightarrow\)’, and ‘\(\neg\)’, there is a logically equivalent proposition \(Q\) which uses only ‘and’, ‘\(\Rightarrow\)’, and ‘\(\neg\)’.

(b) Show that \((P \text{ and } Q)\) is logically equivalent to an expression involving only \(P, Q, \text{ ‘\(\Rightarrow\)’},\) and ‘\(\neg\)’. Conclude, as in (a), that we can get by with just ‘\(\Rightarrow\)’, and ‘\(\neg\)’. 

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15. The *nand* (as in “not and”) operator ’↑’ is defined via the truth table

\[
\begin{array}{ccc}
P & Q & P \uparrow Q \\
T & T & F \\
T & F & T \\
F & T & T \\
F & F & T \\
\end{array}
\]

(a) Show that \( \neg P \) is logically equivalent to \( P \uparrow P \).

(b) Show that \( P \Rightarrow Q \) is logically equivalent to a proposition involving only \( P, Q \) and \( \uparrow \).

(c) Conclude, using the previous problem that any compound proposition built up from the operators ‘or’, ‘and’, ‘⇒’, and ‘¬’ is equivalent to a proposition that uses only the operator ‘↑’. In other words we can get by with just one logical connective!