1. Let $m$ and $n$ be nonzero integers, and let $p$ be a prime. Prove that $v_p(m \cdot n) = v_p(m) + v_p(n)$.

2. Let $m \neq 0$ be an integer and let $p$ be a prime. Suppose $v_p(m) = r$. Prove that we can write $m = p^r u$, where $u$ is an integer with $v_p(u) = 0$.

3. Let $m$ and $n$ be nonzero integers, and let $p$ be a prime. Let’s investigate $v_p(m + n)$, the exponent of $p$ in the prime factorization of $m + n$.
   (a) Prove that $v_p(m + n) \geq \min(v_p(m), v_p(n))$.
   (b) Show that if $v_p(m) \neq v_p(n)$, then $v_p(m + n) = \min(v_p(m), v_p(n))$.
   (c) Give an example where $v_p(m + n) > \min(v_p(m), v_p(n))$. You must specify what $m$, $n$ and $p$ are.

4. Let $n \in \mathbb{Z}^+$, and let $p$ be a prime. Prove that
   \[ v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots \]
   [Hint: recall that $\left\lfloor \frac{n}{p^i} \right\rfloor$ is equal to the number of multiples of $p^i$ between 1 and $n$.]
   For example, $v_3(10!) = \left\lfloor \frac{10}{3} \right\rfloor + \left\lfloor \frac{10}{9} \right\rfloor = 3 + 1 = 4$; i.e., in the prime factorization of $10!$, the exponent of the prime 3 is 4.
   Note that the infinite sum in our expression always reduces to a finite one, since $\left\lfloor \frac{n}{p^i} \right\rfloor$ is 0 for all $i$ with $p^i > n$.

5. Recall that an integer $n$ is a perfect square if $n = a^2$ for some integer $a$.
   (a) Prove that an integer $n$ is a perfect square if and only if $v_p(n)$ is even for all primes $p$.
   (b) Prove that $n$ is a perfect square if and only if $n$ has an odd number of positive divisors.

6. Define an integer $n$ to be a perfect $r$-th power if $n = a^r$ for some integer $a$.
   (a) State and prove necessary and sufficient conditions for $n$ to be a perfect $r$-th power in terms of $v_p(n)$.
   (b) Let $n \in \mathbb{Z}^+$. Suppose that $n$ is not a perfect $r$-th power. Prove that $\sqrt[r]{n}$ (the positive $r$-th root of $n$) is irrational.

7. Suppose $\alpha$ is a real number satisfying $\alpha^4 + \alpha = 10$. Prove that $\alpha$ is irrational. [Hint: $\mathbb{Z}$ is integrally closed!]
8. Let $p$ be a prime. We can extend the definition of $v_p$ to rational numbers as follows: given a nonzero rational $q = \frac{m}{n}$ with $m, n \in \mathbb{Z}$, define $v_p(q) = v_p(m) - v_p(n)$.

(a) Recall that rational numbers can be expressed in many different ways as a quotient of integers. Show that our definition of $v_p$ is well-defined despite this; i.e., show that if $q = \frac{m}{n}$ and $q = \frac{r}{s}$ are two different representations of $q$, then $v_p(m) - v_p(n) = v_p(r) - v_p(s)$.

(b) Let $q$ and $q'$ be nonzero rational numbers, and let $p$ be prime. Prove that $v_p(q \cdot q') = v_p(q) + v_p(q')$. (Note this is an extension of our earlier result about integers).

(c) Recall that $\mathbb{Z} \subset \mathbb{Q}$. Prove that $\mathbb{Z} = \{q \in \mathbb{Q}: v_p(q) \geq 0 \text{ for all primes } p\}$. 